EXCERPTED FROM

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A NEW KIND OF SCIENCE

SECTION 6.7

The Notion of Attractors

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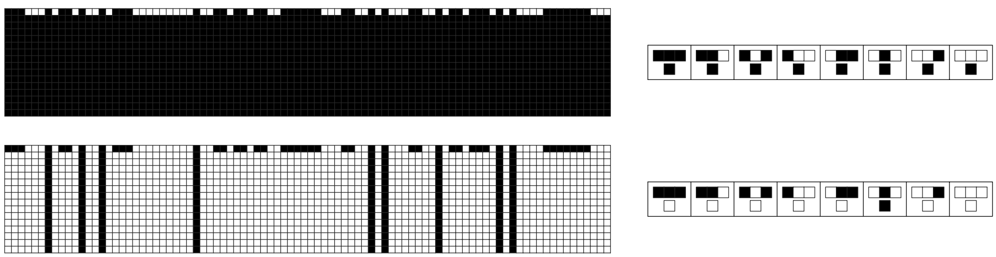
In this chapter we have seen many examples of patterns that can be produced by starting from random initial conditions and then following the evolution of cellular automata for many steps.

But what can be said about the individual configurations of black and white cells that appear at each step? In random initial conditions, absolutely any sequence of black and white cells can be present. But it is a feature of most cellular automata that on subsequent steps the sequences that can be produced become progressively more restricted.

The first picture below shows an extreme example of a class 1 cellular automaton in which after just one step the only sequences that can occur are those that contain only black cells.

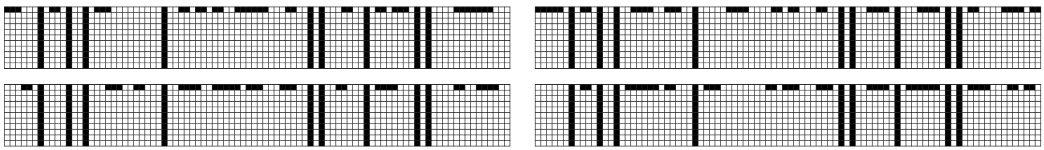


Examples of simple cellular automata that evolve after just one step to attractors in which only certain sequences of black and white cells can occur. In the first case, the sequences that can occur are ones that involve only black cells. In the second case, the sequences are ones in which every black cell is surrounded by white cells. The rules shown are numbers 255 and 4.

The resulting configuration can be thought of as a so-called attractor for the cellular automaton evolution. It does not matter what initial conditions one starts from: one always reaches the same all-black attractor in the end. The situation is somewhat similar to what happens in a mechanical system like a physical pendulum. One can start the pendulum swinging in any configuration, but it will always tend to evolve to the configuration in which it is hanging straight down.

The second picture above shows a class 2 cellular automaton that once again evolves to an attractor after just one step. But now the attractor does not just consist of a single configuration, but instead consists of all configurations in which black cells occur only when they are surrounded on each side by at least one white cell.

The picture below shows that for any particular configuration of this kind, there are in general many different initial conditions that can lead to it. In a mechanical analogy each possible final configuration is like the lowest point in a basin—and a ball started anywhere in the basin will then always roll to that lowest point.



Four different initial conditions that all lead to the same final state in the rule 4 cellular automaton shown on the previous page. The final state can be thought of as one of the possible attractors for the evolution of the cellular automaton; the initial conditions shown then represent different elements in the basin of attraction for this attractor.

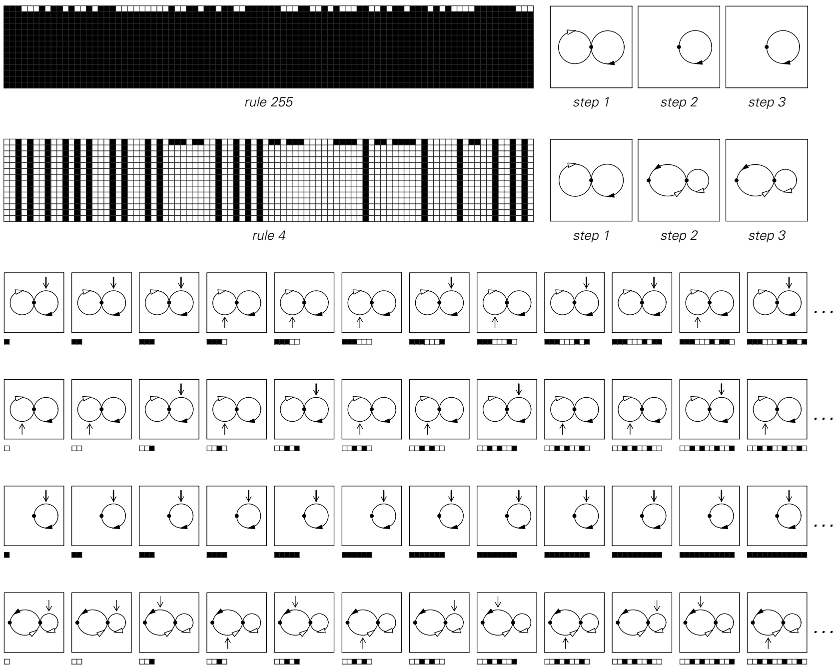
For one-dimensional cellular automata, it turns out that there is a rather compact way to summarize all the possible sequences of black and white cells that can occur at any given step in their evolution.

The basic idea is to construct a network in which each such sequence of black and white cells corresponds to a possible path.

In the pictures at the top of the facing page, the first network in each case represents random initial conditions in which any possible sequence of black and white cells can occur. Starting from the node in the middle, one can go around either the left or the right loop in the network any number of times in any order—representing the fact that black and white cells can appear any number of times in any order.

At step 2 in the rule 255 example on the facing page, however, the network has only one loop—representing the fact that at this step the only sequences which can occur with this rule are ones that consist purely of black cells, just as we saw on the previous page.

The case of rule 4 is slightly more complicated: at step 2, the possible sequences that can occur are now represented by a network with two nodes. Starting at the right-hand node one can go around the loop to the right any number of times, corresponding to sequences of



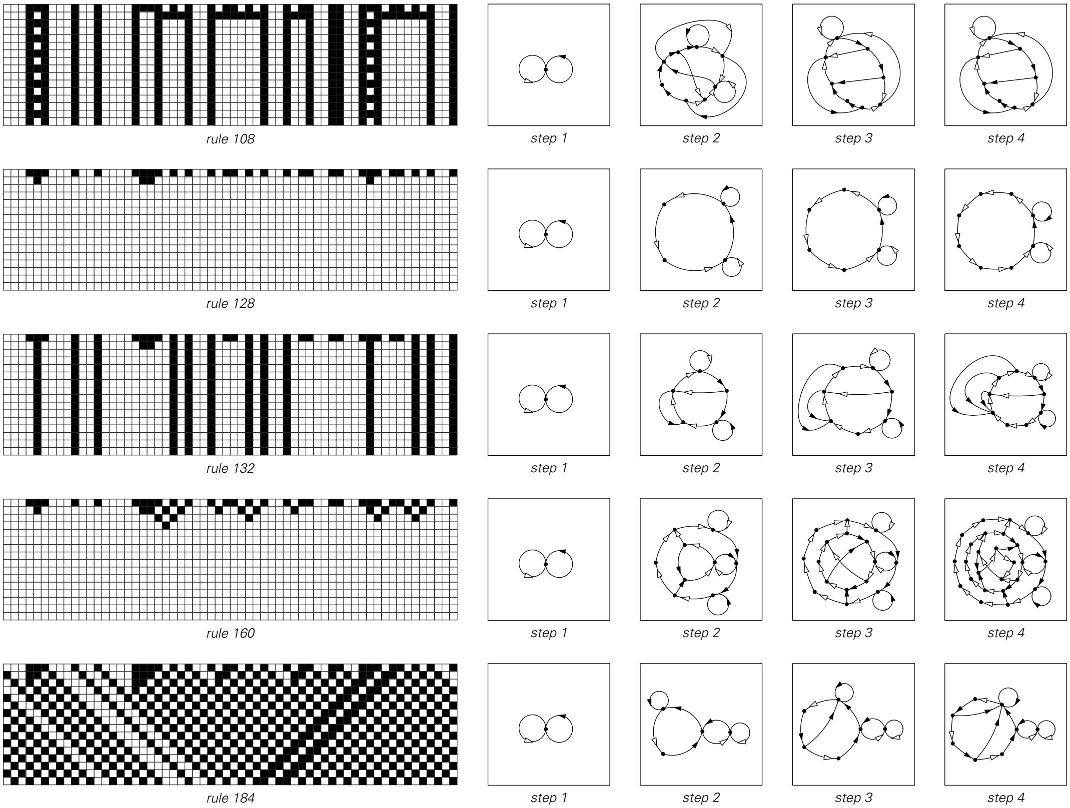
Networks representing possible sequences of black and white cells that can occur at successive steps in the evolution of the two cellular automata shown on the left. In each case the possible sequences correspond to possible paths through the network. Both rules start on step 1 from random initial conditions in which all sequences of black and white cells are allowed. On subsequent steps, rule 255 allows only sequences containing just black cells, while rule 4 allows sequences that contain both black and white cells, but requires that every black cell be surrounded by white cells.

any number of white cells. At any point one can follow the arrow to the left to get a black cell, but the form of the network implies that this black cell must always be followed by at least one white cell.

The pictures on the next page show more examples of class 1 and 2 cellular automata. Unlike in the picture above, these rules do not reach their final states after one step, but instead just progressively evolve towards these states. And in the course of this evolution, the set of sequences that can occur becomes progressively smaller.

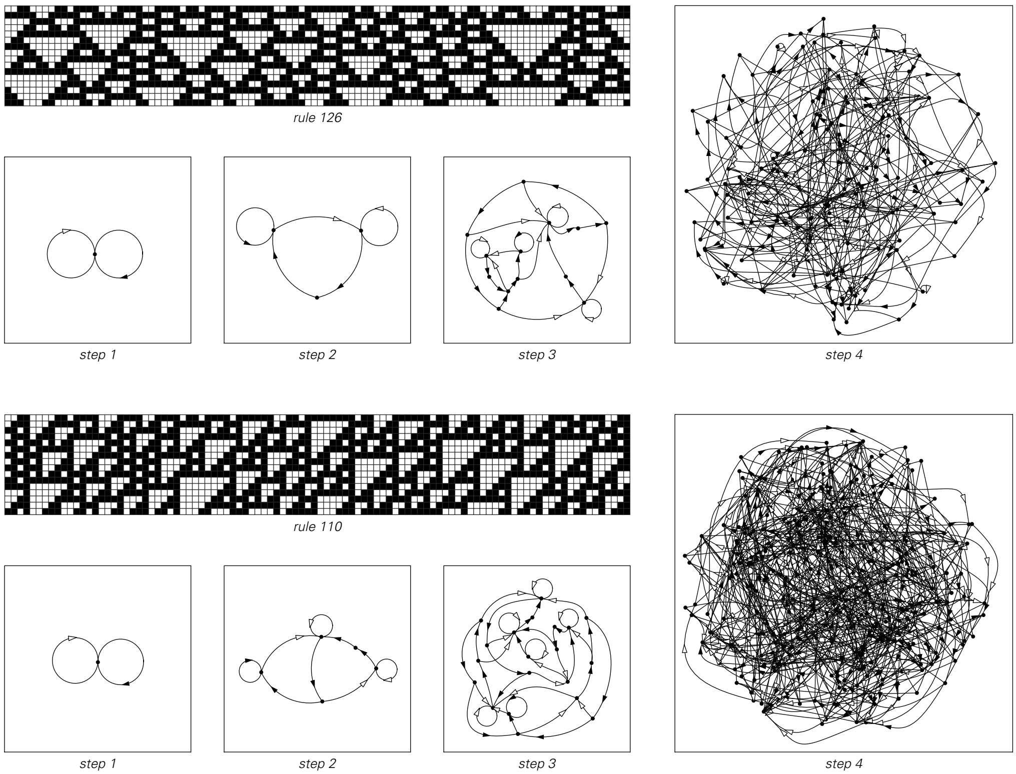
In rule 128, for example, the fact that regions of black shrink by one cell on each side at each step means that any region of black that exists after t steps must have at least t white cells on either side of it.

The networks shown on the next page capture all effects like this. And to do this we see that on successive steps they become somewhat more complicated. But at least for these class 1 and 2 examples, the progression of networks always continues to have a fairly simple form.



Networks representing possible sequences of black and white cells that can occur at successive steps in the evolution of several class 1 and 2 cellular automata. These networks never have more than about t^2 nodes after t steps.

So what happens with class 3 and 4 systems? The pictures on the facing page show a couple of examples. In rule 126, the only effect at step 2 is that black cells can no longer appear on their own: they must always be in groups of two or more. By step 3, it becomes difficult to see any change if one just looks at an explicit picture of the cellular automaton evolution. But from the network, one finds that now an infinite collection of other blocks are forbidden, beginning with the length 12 block . And on later steps, the set of sequences that are allowed rapidly becomes more complicated—as reflected in a rapid increase in the complexity of the corresponding networks.



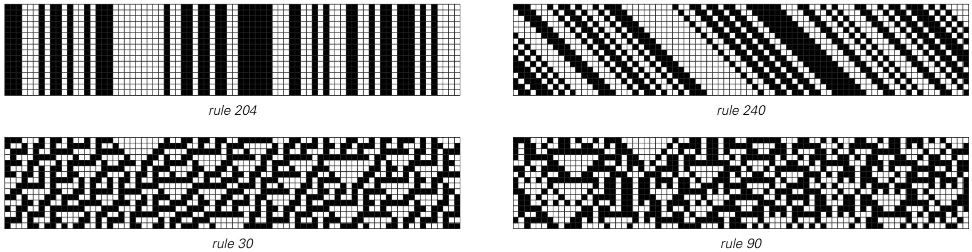
Networks representing possible sequences of black and white cells that can occur at successive steps in the evolution of typical class 3 and 4 cellular automata. The number of nodes in these networks seems to increase at a rate that is at least exponential.

Indeed, this kind of rapid increase in network complexity is a general characteristic of most class 3 and 4 rules. But it turns out that there are a few rules which at first appear to be exceptions.

The pictures at the top of the next page show four different rules that each have the property that if started from initial conditions in which all possible sequences of cells are allowed, these same sequences can all still occur at any subsequent step in the evolution.

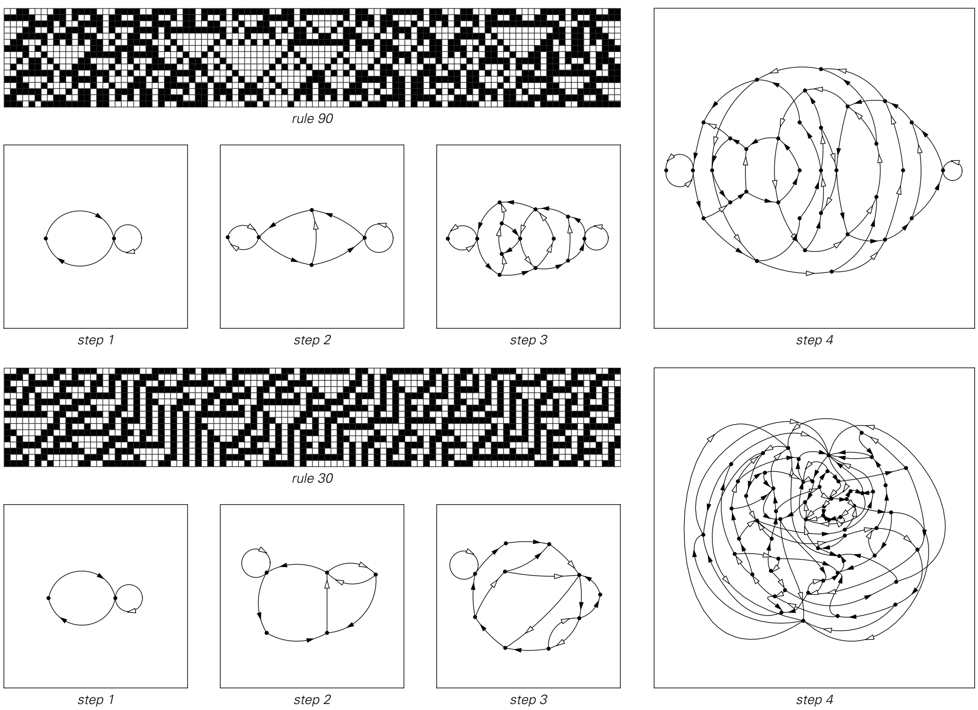
The first two rules that are shown exhibit very simple class 2 behavior. But the last two show typical class 3 behavior.

What is going on, however, is that in a sense the particular initial conditions that allow all possible sequences are special for these rules.



Examples of cellular automata which continue to allow all possible sequences of black and white cells at any step in their evolution. Such cellular automata in effect define what are known as surjective or onto mappings.

And indeed if one starts with almost any other initial conditions—say for example ones that do not allow any pair of black cells together, then as the pictures below illustrate, rapidly increasing complexity in the sets of sequences that are allowed is again observed.



Networks representing possible sequences that can occur in the evolution of the cellular automata at the top of the page, starting from initial conditions in which black cells are only allowed to appear in pairs.